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# Mh4718 Week 9

# Week 9

## 0.1 Solving Differential Equations (Contd.)

### Example 0.1

1. Determine the Taylor expansion of $(1+x)^{\frac{1}{2}}$ around 0.

$$\begin{aligned} f(x) &= (1+x)^{\frac{1}{2}} & \Rightarrow f(0) &= 1 \\ f^{(1)}(x) &= \tfrac{1}{2}(1+x)^{-\frac{1}{2}} & \Rightarrow f^{(1)}(0) &= \tfrac{1}{2} \\ f^{(2)}(x) &= \tfrac{1}{2}(-\tfrac{1}{2})(1+x)^{-\frac{3}{2}} & \Rightarrow f^{(2)}(0) &= \tfrac{1}{2}(-\tfrac{1}{2}) \\ f^{(3)}(x) &= \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})(1+x)^{-\frac{5}{2}} & \Rightarrow f^{(3)}(0) &= \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2}) \end{aligned}$$

Continuing like this we see that the Taylor series will be

$$1 + \tfrac{1}{2}x - \frac{\frac{1}{2}(\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})}{3!}x^3 - \frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{4!}x^4 - \frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{5!}x^5 \ldots$$

2. Use Taylor series to find a solution for the IVP $\dfrac{dy}{dx} = \dfrac{1}{2y}, y(0) = 1$

This differential equation can also be written $\dfrac{dy}{dx} = \frac{1}{2}y^{-1}$. And then we have

$$
\begin{aligned}
y(0) &= 1 \\
y^{(1)}(x) &= \frac{1}{2y} \Rightarrow y^{(1)}(0) = \frac{1}{2y(0)} = \frac{1}{2} \\
y^{(2)}(x) &= -\tfrac{1}{2}y^{-2}y^{(1)} = -\tfrac{1}{2}y^{-2}\tfrac{1}{2}y^{-1} = -(\tfrac{1}{2})(\tfrac{1}{2})y^{(-3)} \\
&\Rightarrow y^{(2)}(0) = -(\tfrac{1}{2})(\tfrac{1}{2}) \\
y^{(3)}(x) &= -(\tfrac{1}{2})(\tfrac{1}{2})(-3)y^{(-4)}y^{(1)} = -(\tfrac{1}{2})(\tfrac{1}{2})(-3)y^{(-4)}\tfrac{1}{2}y^{-1} = -(\tfrac{1}{2})(\tfrac{1}{2})(\tfrac{-3}{2})y^{(-5)} \\
&\Rightarrow y^{(3)}(0) = (\tfrac{1}{2})(\tfrac{1}{2})(\tfrac{3}{2})
\end{aligned}
$$

Continuing like this we get the Taylor series for the solution:

$$
1 + \tfrac{1}{2}x - \frac{\frac{1}{2}(\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})}{3!}x^3 - \frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{4!}x^4 - \frac{\frac{1}{2}(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{5!}x^5 \ldots
$$

From our previous work we recognise this as the Taylor series around 0 for the function $(1+x)^{\frac{1}{2}}$ and we can verify that this is the solution of the IVP.

3. The Taylor expansion around 0 of $x^3 + 1$ is $1 + x^3$.
4. The Taylor expansion around 1 of $x^3+1$ is $2+3(x-1)+3(x-1)^2+(x-1)^3$.
5. Use Taylor series to solve the IVP $\dfrac{dy}{dx} = \dfrac{3y-3}{x}, y(1) = 2$.

$$
\begin{aligned}
y(1) &= 2 \\
y^{(1)}(x) &= \frac{3y-3}{x} \Rightarrow y^{(1)}(1) = \frac{6-3}{1} = 3 \\
y^{(2)}(x) &= \frac{3y^{(1)}x - (3y-3)}{x^2} = \frac{3\frac{3y-3}{x}x - (3y-3)}{x^2} = \frac{6y-6}{x^2} \Rightarrow y^{(2)}(1) = 6 \\
y^{(3)}(x) &= \frac{6y^{(1)}x^2 - (6y-6)2x}{x^4} = \frac{6\frac{3y-3}{x}x^2 - (6y-6)2x}{x^4} = \frac{6yx-6x}{x^4} \\
&\Rightarrow y^{(3)}(1) = 6 \\
y^{(4)}(x) &= \frac{(6y^{(1)}x + 6y - 6)x^4 - (6yx-6x)4x^3}{x^8} = \frac{24yx^4 - 24x^4 - 24yx^4 + 24x^4}{x^8} = 0
\end{aligned}
$$

The Taylor series for the solution is then $2+3(x-1)+3(x-1)^2+(x-1)^3$ which is $x^3+1$.

### 0.1.1 Euler's Method

*Euler's method* uses just the first two terms of the Taylor expansion to estimate the solution to a differential equation.
That is, if $f(x)$ is a solution of $\frac{dy}{dx} = F(x, y)$ with "initial value" $f(x_0)$ the Taylor polynomial of degree 1 around $x_0$ is

$$f(x_0) + f'(x_0)(x - x_0) = f(x_0) + F(x_0, f(x_0))(x - x_0)$$

This will be a very crude estimate for $f(x)$ with error term $f^{(2)}(c)\frac{(x - x_0)^2}{2}$ where $c$ is some point between $x$ and $x_0$.
The only way we can reduce the size of the error is to keep $x - x_0$ small, that is, keep $x$ close to $x_0$.
If we let $x_1 = x_0 + h$ where $h$ is small then

$$f(x_1) \approx f(x_0) + F(x_0, f(x_0)(x_1 - x_0) = f(x_0) + F(x_0, f(x_0))h$$

This gives us an approximation to the solution at a point $x_1$ a small distance away from the initial point.

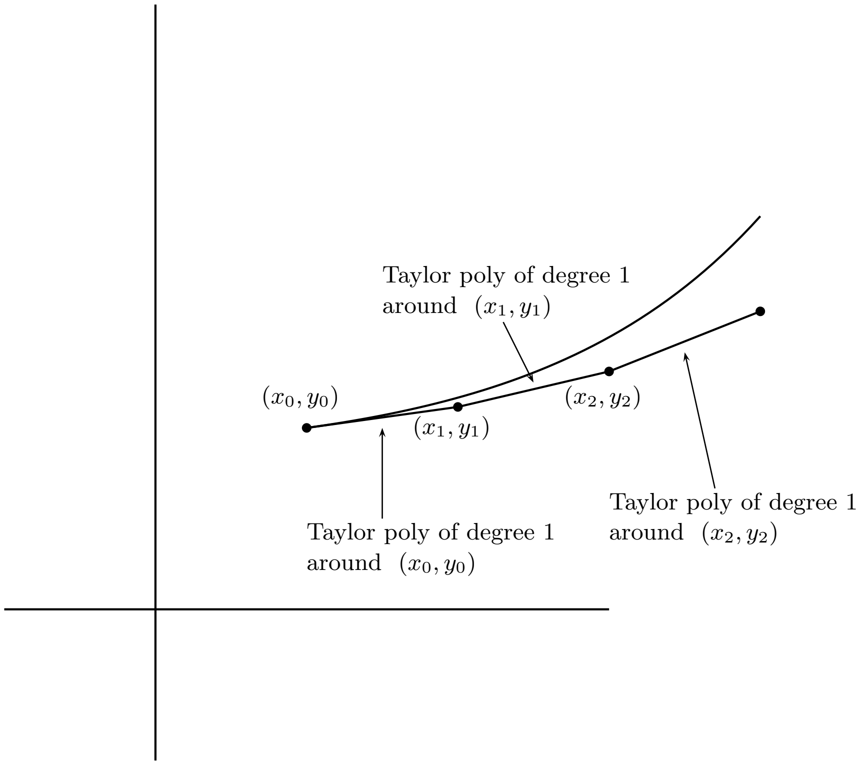
We then use this approximation to generate a Taylor polynomial of degree 1 around the new point $x_1$ in order to estimate $f(x_2)$ where $x_2 = x_1 + h$. That is

$$f(x_2) \approx f_{approx}(x_1) + F(x_1, f_{approx}(x_1))h.$$

Similarly we get:

$$f(x_3) \approx f_{approx}(x_2) + F(x_2, f_{approx}(x_2))h.$$

where $x_3 = x_2 + h$ and so on.
The following diagram illustrates how Euler's method generates an approximation to the value of the solution at $x_i$ by using the value of the Taylor polynomial of degree 1 at $x_{i-1}$. The approximation at $x_i$ is in turn used to generate an approximation at $x_{i+1}$ and so on. Obviously all Taylor polynomials except the first one are based on approximate values of the solution so there is an added error incurred at each step of the method. However, these errors are usually considerably smaller than the error that we would incur if we simply used the Taylor polynomial around $(x_0, y_0)$ only.




## Example 0.2

Consider the differential equation $\dfrac{dy}{dx} = y$. That is $F(x, y) = y$.
We know that $y = e^x$ is a solution of this equation with initial condition $y(0) = 1$.
The following code outputs to a text file the co-ordinates for the graph of the of the approximate solution generated by Euler's method and also the corresponding values of the exact solution $e^x$ over the interval $[0, 4]$.

```
#include <iostream>
#include<cmath>
#include<fstream>
using namespace std;

double F(double x, double y)
{
```

```
    return y;
}
void main()
{
    ofstream fout("euler.txt");
    double h=0.1;
    double ny;
    for(double x=0;x<=4;x+=h)
    {
         ny=y+F(x,y)*h;// Taylor poly of degree 1
         fout<<x<<"\t"<<y<<"\t"<<exp(x)<<endl;
         y=ny
    }
}
```

Euler's method is also known as an order 1 Taylor method since it uses a Taylor polynomial of degree 1 to estimate the value of the solution to a differential equation at a point close to the last point at which it was estimated.
At each step of the method there will be an inaccuracy in the estimate of the next step and the estimate for the next step will include this innacuracy as well as the innaccuracy of the current estimate and so the estimated solution will drift farther from the exact one at each step.

The accuracy at each step can be improved by using a higher order Taylor method i.e. by using a Taylor polynomial of higher degree to generate the new estimates.

That is, instead of

$$\begin{aligned} f_{approx}(x+h) &= f_{approx}(x) + f'_{approx}(x)h \\ &= f_{approx}(x) + F(x, f_{approx}(x))h \end{aligned}$$

we have

$$\begin{aligned} f_{approx}(x+h) &= f_{approx}(x) + f'_{approx}(x)h + f''_{approx}(x)\frac{h^2}{2} \\ &= f_{approx}(x) + F(x, f_{approx}(x))h + f''_{approx}(x)\frac{h^2}{2}. \end{aligned}$$

We can get $f''(x)$ from the differential equation. That is,

$$f'(x) = F(x, f(x)) \Rightarrow f''(x) = \frac{d}{dx}F(x, f(x)).$$

And so, if we let $F1(x) = \dfrac{d}{dx}F(x, f(x))$, the Taylor method of order 2 is:

$$f_{approx}(x+h) = f_{approx}(x) + F(x, f_{approx}(x))h + F1(x, f_{approx}(x))\frac{h^2}{2}.$$

Code for Taylor method of order 2:

```
#include <iostream>
#include <cmath>
#include <iomanip>
#include <fstream>

using namespace std;

double F(double x, double y)
{
    return y; //The differential equation is dy/dx =y
}

double F1(double x,double y)
{
    return y;//In this case d/dx F(x,y) = dy/dx = y
}
void main()
{

    double h = pow(2,-3); //The step size
    double ny;
    for(double x=0; x<=4;x+=h)
    {
        ny=y+F(x,y)*h+F1(x,y)*h*h/2;
        fout<<x<<"\t"<<y<<"\t"<<exp(x)<<endl;
        y=ny;
    }

}
```

Yet higher order Taylor methods can be used but these will require more calculations of the higher order derivatives of the solution function and this can be difficult if the differential equation is complicated.

### 0.1.2 Separation of variables

The technique of separation of variables uses the so-called *substitution rule* for integration. The *substitution rule* is in turn based on the *chain rule* for differentiation.

Recall that, according to the chain rule (assuming that all functions are suitably differentiable), we have

$$\frac{d}{dx}f(u(x)) = \frac{d}{du}f(u)\frac{d}{dx}u(x)$$

## Example 0.3

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$$

Recall also that an indefinite integral is an *anti-derivative* i.e.

$$\int \left(\frac{d}{dx}f(x)\right) \mathrm{d}x = f(x).$$

## Example 0.4

$$\int x^2 \mathrm{d}x = \tfrac{1}{3}x^3 + constant$$

because

$$\frac{d}{dx}(\tfrac{1}{3}x^3 + constant) = x^2.$$

Therefore we can see that

$$\int \left(\frac{d}{du}f(u)\frac{d}{dx}u(x)\right) \mathrm{d}x = f(u(x)).$$

## Example 0.5

$$\int 2x\cos(x^2)\mathrm{d}x = \sin(x).$$

The *substitution rule* is now obvious because

$$\int \left(\frac{d}{du}f(u)\frac{d}{dx}u(x)\right) \mathrm{d}x = f(u(x)) = \int \frac{d}{du}f(u)\mathrm{d}u.$$

Notationally we see that

$$\frac{d}{dx}u(x)\mathrm{d}x$$

in the left hand integral has been replaced by

$$\mathrm{d}u$$

in the right hand integral (as if $\mathrm{d}x$ has been cancelled!)
Thus we get the *substitution rule*

$$\int F(u)\frac{du}{dx}\mathrm{d}x \to \int F(u)\mathrm{d}u.$$

## Example 0.6

$$\int \overset{\frac{du}{dx}}{\overset{\downarrow}{2x}} \cos(\overset{u}{\overset{\downarrow}{x^2}})\mathrm{d}x = \int \cos(u)\mathrm{d}u = \sin(u) = \sin(x^2)$$